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Attenuation of sound in a low Mach number nozzle flow

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This paper examines the energy conversion mechanisms which govern the emission of low frequency sound from an axisymmetric jet pipe of arbitrary nozzle contraction ratio in the case of low Mach number nozzle flow. The incident acoustic energy which escapes from the nozzle is partitioned between two distinct disturbances in the exterior fluid. The first of these is the free-space radiation, whose directivity is equivalent to that produced by monopole and dipole sources. Second, essentially incompressible vortex waves are excited by the shedding of vorticity from the nozzle lip, and may be associated with the large-scale instabilities of the jet. Two linearized theoretical models are discussed. One of these is an exact linear theory in which the boundary of the jet is treated as an unstable vortex sheet. The second assumes that the finite width of the mean shear layer of the real jet cannot be neglected. The analytical results are shown to compare favourably with recent attenuation measurements.

1. Introduction

This paper examines the energy conversion mechanisms involved in the emission of sound from the interior of a jet pipe in the presence of a subsonic nozzle flow. This is particularly relevant to the problem of 'excess' or 'core' noise produced by unsteady combustion and turbine blading in the jet pipe of an aeroengine. It is also of interest in connexion with the energy balance associated with the generation of resonant oscillations in the pipe and in musical instruments such as the flute.

According to experiments of Crow (1972) and of Gerend, Kumasaka & Roundhill (1973), upstream-generated sound is significantly amplified by passage through the jet at subsonic velocities, the additional radiation being attributed to the excitation of instability waves of the jet. This conclusion has been challenged by Moore (1977) and by Bechert, Michel & Pfizenmaier (1977), who pointed out that it was based on measurements of the acoustic intensity at a single far-field location. In a series of carefully conducted experiments Moore demonstrated that, over a wide frequency range and for jet Mach numbers lying between 0·1 and 0·9, there is no significant overall radiation from the instability mode at the excitation frequency.

This question was investigated by Bechert *et al.* (1977) using an acoustic tone generated within the jet pipe by means of a system of matched loudspeakers. Although their experiment was confined to the case of a cold subsonic jet, which precluded a strict comparison with the earlier work, the absence of amplification at the tonal frequency was confirmed. Moreover, at sufficiently low acoustic frequencies, specifically for Helmholtz numbers ka less than unity, k being the acoustic wavenumber and a the nozzle-exit radius, a considerable attenuation of the tone was observed during its emission through the nozzle flow into free space, an effect also reported by Moore (1977), and amounted to 15 dB or more for $ka \sim 0{\cdot}2$. A high level of tonal excitation is known to bring about an overall increase in the broad-band noise produced by the jet (Bechert & Pfizenmaier 1975*a*; Moore 1977), but Bechert *et al.* (1977) were able to show that this additional radiation in no way compensates for the strong attenuation of the tone, the relationship between the broad-band amplification and the excitation amplitude being essentially nonlinear.

Munt (1977) has described in detail a linearized analytical theory of the radiation of sound from a circular cylindrical pipe in the presence of a subsonic nozzle flow. The jet shear layer was approximated by an infinitely thin cylindrical vortex sheet, and free-space radiation directivities calculated from this model were shown to be in excellent agreement with field shape data obtained by Pinker & Bryce (1976) using a jet pipe with a conical nozzle. This led Bechert *et al.* (1977) to suggest that the same theory could well account for the attenuation observed in their experiment at low frequencies. In this paper we shall verify that this is indeed the case. No direct use will be made of Munt's formulae, however, since, although valid over a wide range of conditions, they offer no insight into the nature of the physical mechanisms which are called into play during the passage of an acoustic disturbance through the nozzle.

The interaction of an acoustic tone with low Mach number nozzle flow has been studied in relation to laminar–turbulent transition in a separated boundary layer. Brown (1935) and the experiments of Freymuth (1966) indicate that the influence of the sound on the free shear layer of the jet is restricted to the region close to the nozzle lip. Bechert & Pfizenmaier (1975*b*) examined the nature of the flow near the lip, and concluded that at sufficiently small Strouhal numbers based on boundary-layer width, the disturbed flow leaves the trailing edge tangentially, in accordance with the Kutta–Joukowski hypothesis. We shall argue below that an attenuation of the acoustic field is necessary in order to energize the essentially incompressible, unsteady flow associated with the vorticity that must be shed from the lip to satisfy the Kutta condition. This may involve the growth of spatial instabilities of the jet, and in this case the attenuation may be regarded as being necessary to maintain the corresponding large-scale 'coherent structures'. Of course, shed vorticity and instability waves are known to produce sound by their subsequent interaction with the nozzle, but at low frequencies the radiated sound power is of order M_J^2 relative to the power loss from the incident sound wave, M_J being the Mach number of the jet. This is accordingly a situation in which the production of aerodynamic quadrupole sources (Lighthill 1952), in the form of initially organized vortical disturbances, results in an overall *reduction* in the acoustic energy!

All available theories of jet–acoustic interaction (e.g. Crighton 1972; Savkar 1975; Munt 1977) employ a vortex-sheet representation of the free shear layer and impose the Kutta condition. The Strouhal numbers of interest in the present discussion are sufficiently small to justify the application of this condition. However, the experiments of Pinker & Bryce (1976) and the results reported by Savkar (1975) indicate that there is no significant excitation of the instability mode for a cold jet operating at low subsonic Mach numbers. This suggests that it may be necessary to take account of the finite width of the mean shear layer, and indeed it may be argued that Pinker & Bryce's experimental results reveal that close to the nozzle lip the radial length scale of the unsteady shed vorticity is much smaller than that of the shear layer.

In this paper the attenuation of the sound will be discussed in terms of Lighthill's (1952) acoustic-analogy theory of aerodynamic sound by means of the formulation proposed by the author (Howe 1975). It will be assumed that the acoustic wavelength is large compared with the radius of the jet pipe, and this will enable the analysis to take account of an arbitrary contraction in the cross-sectional area of the pipe at the nozzle. The general problem is formulated in § 2 and the characteristics of the free-space radiation field are deduced. The mechanism of energy transfer to the essentially incompressible vortex motions of the jet is described in § 3; specific details are given for an exterior shear flow modelled by a vortex sheet, and also for an approximate treatment of the case of finite shear-layer width (§ 4). The predictions of the analysis are discussed in relation to the experiments of Pinker & Bryce (1976) and Bechert *et al.* (1977). Various analytical results are collected together in an appendix.

2. The radiation of internally generated sound from a low Mach number nozzle flow

2.1. *Formulation of the problem*

An axisymmetric air jet of density ρ_1 and sound speed c_1 exhausts from a jet pipe of cross-sectional area $\mathscr{A}$ through a nozzle of area A into a stationary ambient medium of density and sound speed respectively equal to ρ_0 and c_0 (figure 1). The Mach number of the flow is taken to be sufficiently small that variations in ρ_1 and c_1 may be neglected. This will be the case if the steady upstream flow velocity U and the nozzle exit velocity U_J satisfy $M^2, M_J^2 \ll 1$, where Mach numbers M and M_J are defined by

$$M = U/c_1, \quad M_J = U_J/c_1. \tag{2.1}$$

Dissipation processes will also be neglected, so that for uniform upstream conditions the flow is homentropic, although there may be a variation in the specific entropy s across the mean shear layer of the jet.

A plane harmonic sound wave is incident on the nozzle exit from within the jet pipe. It is required to determine the relation between the flux W_T, say, of acoustic energy through the nozzle, i.e. through the control surface Σ located just upstream of the contraction, and the total acoustic power W_F radiated into the ambient medium. Let p_I denote the amplitude of the incident wave, such that in the upstream region the incident pressure perturbation is given by the real part of

$$p = p_I \exp\left[i\left(\frac{k_1 x_1}{1+M} - \omega t\right)\right]. \tag{2.2}$$

In this expression ω is the radian frequency, $k_1 = \omega/c_1$, t is the time, and the positive direction of the x_1 axis of a rectangular co-ordinate system (x_1, x_2, x_3) is parallel to the mean flow, the origin being located in the centre of the nozzle exit plane.

The velocity $\mathbf{U}$ of the mean flow is a function of position both within the nozzle and in the exterior fluid, and in this case the Lighthill (1952) acoustic-analogy theory of aerodynamic sound assumes a convenient form when the stagnation enthalpy B, rather than the pressure, is taken as the fundamental acoustic variable. The stagnation enthalpy is given in terms of the velocity $\mathbf{v}$ and the specific enthalpy h by

$$B = h + \tfrac{1}{2}\mathbf{v}^2. \tag{2.3}$$

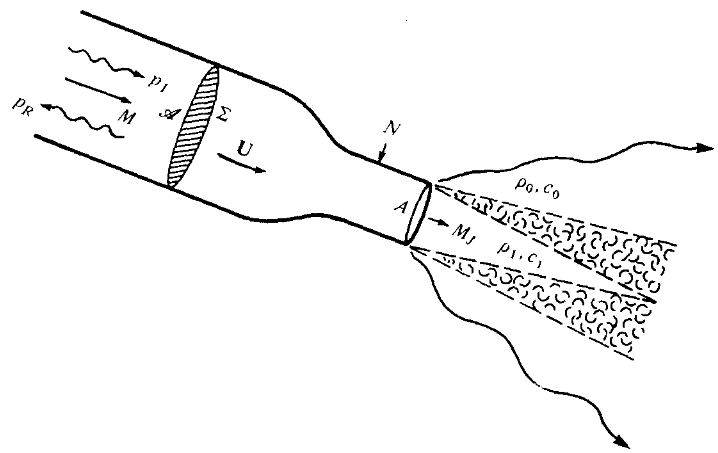

FIGURE 1. Schematic illustration of the configuration considered in the analysis of the emission of low frequency sound from a jet pipe in the presence of a mean nozzle flow.

In the absence of dissipative processes the inhomogeneous wave equation of the acoustic-analogy theory becomes

$$\left\{\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right)+\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\nabla-\nabla^2\right\}B = \operatorname{div}\boldsymbol{\chi}-\frac{1}{c^2}\frac{D\mathbf{v}}{Dt}\cdot\boldsymbol{\chi}, \tag{2.4}$$

where

$$\boldsymbol{\chi} = \boldsymbol{\omega}\wedge\mathbf{v}-T\nabla s, \tag{2.5}$$

$D/Dt = \partial/\partial t+\mathbf{v}\cdot\partial/\partial\mathbf{x}$, $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ is the vorticity, c is the local sound speed, and T is the temperature (Howe 1975).

The terms on the right of (2.4) vanish identically except in the shear layer of the jet. The fluid is homentropic in the ambient medium and within and upstream of the potential core of the jet. In those regions the pressure is a function of the density alone, and the specific enthalpy h may be identified with $\int dp/\rho$. Similarly, Crocco's form of the momentum equation

$$\partial\mathbf{v}/\partial t+\nabla B = -\boldsymbol{\chi} \tag{2.6}$$

(Liepmann & Roshko 1957, p. 193) reduces to the statement that the flow is irrotational outside the jet mixing region, with

$$B = B_n-\partial\phi/\partial t \quad (n = 0, 1). \tag{2.7}$$

Here ϕ is the perturbation velocity potential, and B_n takes respectively constant values B_0, B_1, say, in the ambient medium and in the potential region of the jet. In free space the acoustic pressure p is given by

$$p/\rho_0 = B', \tag{2.8}$$

where $B' = B-B_0$.

It follows from these remarks that, when the mean flow is disturbed by the incident wave (2.2), a linearized description of the subsequent motion in the potential regions is obtained by setting the variable coefficients of the wave operator on the left of (2.4) equal to their local undisturbed mean values. When terms $O(M_J^2)$ relative to unity are also discarded the propagation of small disturbances may be taken to be described by the convected wave equation

$$\left\{\frac{1}{c_1^2}\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right)^2-\frac{\partial^2}{\partial x_j^2}\right\}B=0 \tag{2.9}$$

in the potential region of the jet, and by

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x_j^2}\right)B=0 \tag{2.10}$$

in the ambient medium.

2.2. *Energy flux within the jet pipe*

The flux of acoustic energy through the control surface Σ (figure 1) into the nozzle may be calculated from the general formula

$$W_T=\mathscr{A}(\rho_1\langle uB'\rangle+U\langle\rho'B'\rangle) \tag{2.11}$$

given by Landau & Lifshitz (1959, §§6, 64). The angle brackets denote an average over a wave period $2\pi/\omega$, u is the perturbation velocity, which at Σ is parallel to the x_1 axis, and ρ' is the perturbation density.

Equation (2.3) may be used to express the incident wave (2.2) in the form

$$B_I'=B_I\exp\left[i\left(\frac{k_1x_1}{1+M}-\omega t\right)\right],\quad B_I=(1+M)\frac{p_I}{\rho_1}. \tag{2.12}$$

Let R be a reflexion coefficient such that upstream of the nozzle contraction the total perturbation stagnation enthalpy is given by the real part of

$$B'=B_I\left\{\exp\left[i\frac{k_1x_1}{(1+M)}\right]+R\exp\left[\frac{-ik_1x_1}{(1-M)}\right]\right\}\exp(-i\omega t). \tag{2.13}$$

Equations (2.6) and (2.13), and the adiabatic relation between density and pressure may now be used to express the energy flux (2.11) in the form

$$W_T=W_0(1-|R|^2), \tag{2.14}$$

where $W_0\equiv\mathscr{A}\rho_1|B_I|^2/2c_1$ is the power flux of the incident wave (2.2). This result can be shown to coincide with Blokhintsev's (1946) formula

$$W_T=\frac{\mathscr{A}}{2\rho_1c_1}\left\{(1+M)^2|p_I|^2-(1-M)^2|p_R|^2\right\}, \tag{2.15}$$

where p_R is the amplitude of the reflected pressure perturbation. The reflexion coefficient R is determined by the exterior flow properties of the jet contained within the aerodynamic source vector $\boldsymbol{\chi}$, and will be discussed in §3.

2.3. *The free-space radiation*

The characteristic acoustic wavelength is assumed to be large compared with the length scale of the nozzle and this, together with the low Mach number restriction, enables the effect of fluid compressibility in the nozzle to be neglected in a first approximation. Broad-band fluctuations in the flow produced by nonlinear fluctuations in the aerodynamic source vector $\boldsymbol{\chi}$ of (2.4) are also ignored. This is justified by the experiments of Bechert *et al.* (1977), which reveal negligible amplification of the broad-band radiation for moderate amplitude tonal excitation. Thus only that component of $\boldsymbol{\chi}$ which is directly proportional to the incident wave need be retained.

These approximations permit the replacement of the propagation operator in the acoustic analogy (2.4) by its standard convected far-field form (2.9) [reducing to (2.10) in free space], and also allow the second aerodynamic source term on the right-hand side to be discarded, yielding

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x_j}\right)^2-\frac{\partial^2}{\partial x_j^2}\right\}B=\frac{\partial \chi_j}{\partial x_j} \tag{2.16}$$

(cf. Howe 1975).

A formal representation of the solution of this equation in free space may be obtained by making use of the 'advanced potential' Green's function $G(\mathbf{x},\mathbf{y},t,\tau)$ which satisfies

$$\left\{\frac{1}{c^2}\left(\frac{\partial}{\partial \tau}+U_j\frac{\partial}{\partial y_j}\right)^2-\frac{\partial^2}{\partial y_j^2}\right\}G=\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau), \tag{2.17}$$

the condition of vanishing normal derivative on the rigid surface of the nozzle, and corresponds to an implosive sink at $(\mathbf{x},t)$ which vanishes for $\tau > t$. In the absence of mean flow the form of $G(\mathbf{x},\mathbf{y},t,\tau)$ has been given by Ffowcs Williams & Howe (1975) in the compact approximation in which the acoustic wavelength is large compared with the length scale of the nozzle; the modification required in the present discussion is outlined in the appendix.

Equation (2.16) is solved by applying Kirchoff's procedure to (2.16) and (2.17) (see, for example, Stratton 1941, chap. 8), and for an observer located in free space we find

$$p/\rho_0\equiv B(\mathbf{x},t)=-\int\boldsymbol{\chi}\cdot\frac{\partial G}{\partial \mathbf{y}}\,d^3\mathbf{y}\,d\tau-\oint_\Sigma\left\{\left(G\frac{\partial B}{\partial y_1}-B\frac{\partial G}{\partial y_1}\right)+\frac{M}{c_1}\left(B\frac{DG}{D\tau}-G\frac{DB}{D\tau}\right)\right\}dy_2\,dy_3\,d\tau, \tag{2.18}$$

where the surface integral is taken over the control surface Σ of figure 1 and

$$D/D\tau=\partial/\partial\tau+U\partial/\partial y_1.$$

In obtaining this result the momentum equation (2.6) has been used to eliminate contributions from surface integrals over the rigid nozzle. The volume integral is restricted to the shear layer of the jet, where $\boldsymbol{\chi}$ is non-zero, and accounts for the sound produced by vorticity and entropy fluctuations induced by the incident sound wave (2.2).

The contribution B_Σ, say, from the surface integral in (2.18) is evaluated by noting that the radiation condition ensures that only the component of B corresponding to the incident wave (2.2) need be considered. This contribution may be deduced from the representation (A 3) of $G(\mathbf{x},\mathbf{y},t,\tau)$ given in the appendix, and is found to be

$$B_\Sigma(\mathbf{x},t)=\frac{-ik_1\mathscr{A}B_I\exp(-i\omega[t])}{2\pi|\mathbf{x}|}, \tag{2.19}$$

where [ ] denotes evaluation at the retarded time $t-|\mathbf{x}|/c_0$.

Similarly, using (A 3) and noting that $\boldsymbol{\chi}$ must be an axisymmetric function of position, the volume integral B_χ of (2.18) becomes

$$B_\chi(\mathbf{x},t)=\frac{ik_1\left(1-\dfrac{c_1}{c_0}\dfrac{A}{\mathscr{A}}\cos\theta\right)}{4\pi|\mathbf{x}|}\int\left[\boldsymbol{\chi}\cdot\frac{\partial F_A}{\partial \mathbf{y}}\right]d^3\mathbf{y}+\frac{ik_1\left(\dfrac{c_1}{c_0}\right)\cos\theta}{4\pi|\mathbf{x}|}\int[\chi_1]\,d^3\mathbf{y}, \tag{2.20}$$

where θ is the angle between the $+x_1$ axis and the observer direction and the function $F_A(\mathbf{y})$ is the potential of incompressible, ideal flow from the nozzle which has unit velocity in the $+x_1$ direction upstream of the nozzle contraction. Actually we have neglected a term in (2.20) which is $O(M)$ smaller, M being the Mach number of the upstream flow. This is a valid approximation for large values of the area ratio $\mathscr{A}/A$, and in discussing experiments of Bechert *et al.* (1977) we shall be concerned principally with a case in which the nozzle exit Mach number $M_J = 0{\cdot}3$ and $M \simeq 0{\cdot}04$.

Combining (2.19) and (2.20), it follows that in the ambient medium

$$\frac{p}{\rho_0} = \frac{-ik_1\mathscr{A}}{2\pi|\mathbf{x}|}\left\{B_I \mathrm{e}^{-i\omega[t]} - \frac{\left(1-\dfrac{c_1}{c_0}\dfrac{A}{\mathscr{A}}\cos\theta\right)}{2\mathscr{A}}\int\left[\boldsymbol{\chi}\cdot\frac{\partial F_A}{\partial y}\right]d^3\mathbf{y} - \frac{ic_1\cos\theta}{2c_0}\int[\chi_1]\,d^3\mathbf{y}\right\}. \tag{2.21}$$

A complete specification of the radiation depends on the distribution $\boldsymbol{\chi}$ of the aerodynamic sources, which is the subject of the next section, although it may be anticipated from the definition (2.5) that the contribution from the terms in $\boldsymbol{\chi}$ in the brace brackets in (2.21) is $O(M_0)B_I$, where $M_0 = U_J/c_0$. The expansion of (2.21) in powers of $M_0\cos\theta$ and the rejection of terms $O(M_0^2)$ relative to unity shows that in the leading approximation the radiation pattern is equivalent to that produced by a monopole–dipole combination, the axis of the dipole being perpendicular to the nozzle exit plane.

3. The flux of energy through the nozzle

3.1. *The reflexion coefficient*

The reflexion coefficient R, which determines the energy flux through the nozzle via (2.14), will be calculated by a procedure in which the acoustic representation (2.13) valid upstream of the control surface Σ is matched with a low frequency representation of the flow in the nozzle. By hypothesis, points near Σ of axial location x_1 are well within an acoustic wavelength of the nozzle exit (so that $k_1x_1 \ll 1$). At such points the right-hand side of (2.13) may be expanded in powers of k_1x_1:

$$B' = B_I\{(1+R)+ik_1x_1(1-R)+\ldots\}\exp(-i\omega t), \tag{3.1}$$

in which terms $O(k_1x_1M)$ relative to unity have been discarded, M being the upstream flow Mach number.

The terms shown explicitly in (3.1) are trivial solutions of the Laplace equation for irrotational incompressible flow, and constitute the upstream limiting form of the corresponding incompressible approximation to the flow in the nozzle. In the potential region of the jet within the nozzle [where (2.7) is valid], this incompressible approximation may be set in the form

$$B' = B_I\{\alpha+\beta(F_A(\mathbf{x})+F_J(\mathbf{x}))\}\exp(-i\omega t), \tag{3.2}$$

where α and β are constant, $F_A(\mathbf{x})$ is the potential function introduced in § 2 which describes axisymmetric flow from the nozzle in the absence of the jet, and $F_J(\mathbf{x})$ represents the correction function required to account for the 'back-reaction' of the exterior jet flow. This back-reaction is produced by the shear flow fluctuations $\boldsymbol{\chi}$, and is given by the *causal* solution of (2.16) in the incompressible limit:

$$-\partial^2 B_J/\partial x_j^2 = \partial\chi_j/\partial x_j, \tag{3.3}$$

where

$$B_J = \beta B_I F_J(\mathbf{x}) \exp(-i\omega t). \tag{3.4}$$

In order to perform the matching of (3.2) with the near-field terms of (3.1) it is necessary to determine the limiting form of B_J at Σ. This can be done by making use of the corresponding limiting form of the Green's function $\mathscr{G}(\mathbf{x}, \mathbf{y})$ which satisfies

$$-\partial^2\mathscr{G}/\partial x_j^2 = \delta(\mathbf{x}-\mathbf{y}) \tag{3.5}$$

and the condition of vanishing normal derivative on the rigid surface of the nozzle. When the source point $\mathbf{y}$ is within or downstream of the nozzle and the observer location $\mathbf{x}$ is in the flow upstream of the contraction, it follows from a simple application of the reciprocal theorem (Rayleigh 1945, § 294) that we may take

$$\mathscr{G}(\mathbf{x}, \mathbf{y}) = -\mathscr{A}^{-1}F_A(\mathbf{y}), \tag{3.6}$$

since in the reciprocal problem this would characterize the potential close to and downstream of the nozzle exit of a unit point source located at the upstream point $\mathbf{x}$.

Hence, forming the convolution product of $\mathscr{G}$ and $\partial\chi_j/\partial x_j$, we have in the upstream region

$$B_J \to \frac{1}{\mathscr{A}}\int \boldsymbol{\chi}\cdot\frac{\partial F_A}{\partial \mathbf{y}}\, d^3\mathbf{y}, \tag{3.7}$$

which is independent of the upstream location $\mathbf{x}$. This result expresses the fact that, in the absence of compressibility, the shear-layer dipole source $\boldsymbol{\chi}$ is incapable of inducing a net axial flux within the nozzle. To be sure, the existence of such a flux would require the dipole to be of infinite strength in order to impart a finite acceleration to the mass of fluid within the semi-infinite jet pipe. The details of the fluctuation velocity upstream of the nozzle exit are therefore described by the potential $F_A(\mathbf{x})$ of (3.2). In the linearized approximation $\boldsymbol{\chi}$ must be proportional to this velocity, i.e. to $\beta B_I \exp(-i\omega t)$. This implies that the upstream limiting value (3.7) may be interpreted in terms of a complex 'hydrodynamic end-correction'

$$l_H = \gamma + i\delta, \tag{3.8}$$

which is determined by the incompressible properties of the exterior shear flow, and is such that upstream of the contraction

$$B_J = -\beta B_I(\gamma+i\delta)\exp(-i\omega t) \equiv \frac{1}{\mathscr{A}}\int \boldsymbol{\chi}\cdot\frac{\partial F_A}{\partial \mathbf{y}}\, d^3\mathbf{y}, \tag{3.9}$$

where real parts are to be taken.

The constant α in (3.2) accounts for the possibility of there being a periodic variation in the fluid pressure which is uniform throughout the whole of the incompressible nozzle flow region, and must therefore be associated with the monopole component of the free-space radiation (2.21) (see below). Let this monopole be represented by

$$\frac{p}{\rho_0} = \frac{\Phi_m}{|\mathbf{x}|}\exp\left[-i\omega\left(t-\frac{|\mathbf{x}|}{c_0}\right)\right], \tag{3.10}$$

for large $|\mathbf{x}|$. At distances $|\mathbf{x}|$ well within a wavelength of the nozzle but large compared with the nozzle exit radius a, this expression may be expanded in powers of the retarded time:

$$\frac{p}{\rho_0} = \frac{\Phi_m}{|\mathbf{x}|}\exp(-i\omega t) + ik_0\Phi_m\exp(-i\omega t) + \dots, \tag{3.11}$$

where $k_0 = \omega/c_0$. The first term on the right-hand side describes an unsteady incompressible *volume* flux through a spherical control surface centred on the nozzle exit which, by continuity, must equal that produced by the potential $F_A(\mathbf{x})$ of (3.2). Noting that $\partial F_A/\partial x_1 \to 1$ as $x_1 \to -\infty$ within the jet pipe, we therefore have

$$\beta B_I \mathscr{A} = -4\pi\Phi_m. \tag{3.12}$$

The second term on the right of (3.11) accounts for a uniform fluctuation in pressure which matches that within the nozzle and potential core provided that

$$\alpha\rho_1 B_I = i\rho_0 k_0 \Phi_m. \tag{3.13}$$

Eliminating Φ_m from these relations, we find

$$\alpha = -\frac{i\beta}{4\pi}\left(\frac{\rho_0}{\rho_1}\right) k_0 \mathscr{A}. \tag{3.14}$$

We are now in a position to match corresponding terms of the representations (3.1) and (3.2) of the acoustic field. To do this first observe that upstream of the contraction $F_A(\mathbf{x}) \simeq x_1 - \lambda$ (see appendix), where the added length λ is real and is proportional to the sum of the Helmholtz organ-pipe end correction (Rayleigh 1945, chap. 16) and a component arising from the increased resistance to flow produced by the contraction.

Letting $x_1 \to -\infty$ in (3.2) and identifying terms in the resulting expression with corresponding members of (3.1), we obtain

$$\beta = ik_1(1-R), \tag{3.15a}$$

$$\beta\{\Lambda + i(\delta + k_0\mathscr{A}\rho_0/4\pi\rho_1)\} = -(1+R), \tag{3.15b}$$

use having been made of (3.9) and (3.14), and where we have set

$$\Lambda = \lambda + \gamma. \tag{3.16}$$

Solving for R we find

$$R = -\frac{1-k_1(\delta + k_0\mathscr{A}\rho_0/4\pi\rho_1 - i\Lambda)}{1+k_1(\delta + k_0\mathscr{A}\rho_0/4\pi\rho_1 - i\Lambda)}, \tag{3.17}$$

and using this expression in (2.14), it follows that the flux of energy from the nozzle can be expressed in the form

$$W_T = \frac{W_0\left(\dfrac{\mathscr{A}}{A}\right)\left\{\dfrac{4k_1\delta A}{\mathscr{A}} + \left(\dfrac{\rho_0 c_0}{\rho_1 c_1}\right)(k_0 a)^2\right\}}{\left(1+\dfrac{\mathscr{A}}{4A}\left[\dfrac{4k_1\delta A}{\mathscr{A}} + \left(\dfrac{\rho_0 c_0}{\rho_1 c_1}\right)(k_0 a)^2\right]\right)^2 + (k_1\Lambda)^2}. \tag{3.18}$$

This result shows that the flux of energy into the nozzle from the incident sound wave (2.2) is determined by two factors, corresponding to each of the terms in the brace brackets of the numerator. The first depends on the imaginary part of the hydrodynamic end correction l_H, and is entirely a feature of the incompressible properties of the exterior jet flow. The second, proportional to $(k_0 a)^2$, arises from the monopole component of the sound radiated into the ambient medium. There is no explicit contribution from the dipole component of the radiation (2.21) because it is automatically contained within the hydrodynamic term of (3.18).

3.2. *The mechanism of hydrodynamic attenuation*

An appreciation of the mechanism by which the exterior jet flow extracts energy from the acoustic field may be obtained from a consideration of the contribution of the back-reaction B_J of the shear flow to the general energy flux formula (2.11), and we shall do this before proceeding to applications of the above results to specific modellings of the shear flow.

In the present approximation only the first ('incompressible') term in the parentheses of (2.11) need be retained. When the pulsatile nozzle flow is normalized with respect to the coefficient β of (3.2) it follows from our earlier discussion that the axial perturbation velocity u depends only on the geometrical configuration of the nozzle [i.e. on $F_A(\mathbf{x})$] and not on the properties of the exterior flow. Hence (2.11) implies that the exterior shear flow induces an additional energy flux W_J, say, through the nozzle, where

$$W_J = \mathscr{A}\rho_1\langle uB_J\rangle. \tag{3.19}$$

Now $\partial F_A/\partial x_1 \simeq 1$ upstream of the nozzle contraction, so that the local fluctuating velocity $\mathbf{v}_A$ that would be produced by the acoustic field in the *exterior* fluid in the absence of the jet is just

$$\mathbf{v}_A = u\nabla F_A. \tag{3.20}$$

Equations (3.9) and (3.19) therefore show that

$$W_J = \int \mathbf{v}_A\cdot\rho_1\boldsymbol{\chi}\,d^3\mathbf{y} \equiv \int \mathbf{v}_A\cdot(\rho_1\boldsymbol{\omega}\wedge\mathbf{v}-\rho_1 T\nabla s)\,d^3\mathbf{y}, \tag{3.21}$$

which states that the power flux through the nozzle induced by the essentially incompressible properties of the exterior flow is proportional to the rate of working of the aerodynamic dipole $\boldsymbol{\chi}$ in the acoustic component $\mathbf{v}_A$ of the fluctuating velocity field. Note that the 'lift' experienced by a vortex element is equal to $-\boldsymbol{\omega}\wedge\mathbf{v}$ per unit mass, and the specific 'inertia' force $T\nabla s$ is equal to $-p\nabla(1/\rho)$ in the absence of dissipation, and represents the reaction of a fluid particle when accelerated in an environment of different density.

A particularly illuminating form of (3.21) emerges in the case of uniform mean density. The 'inertia' force vanishes identically, and in the incompressible limit

$$(\boldsymbol{\omega}\wedge\mathbf{v})_i = \frac{\partial}{\partial x_j}(v_i v_j) - \frac{\partial}{\partial x_i}(\tfrac{1}{2}v^2). \tag{3.22}$$

Integrate (3.21) by parts, and observe that there is no contribution from the resulting integral over the surface of the nozzle, to obtain

$$W_J = -\int \epsilon_{ij}\rho_0 v_i v_j\,d^3\mathbf{y}, \tag{3.23}$$

where

$$\epsilon_{ij} = \tfrac{1}{2}(\partial v_{Ai}/\partial x_j + \partial v_{Aj}/\partial x_i) \tag{3.24}$$

is the tensor rate of strain produced by the acoustic component of the fluctuating nozzle flow. The velocity v_i in (3.22) and (3.23) may be regarded as the total velocity minus the acoustic component $\mathbf{v}_A$, since $\operatorname{curl}\mathbf{v}_A \equiv 0$ and the contribution $-\boldsymbol{\omega}\wedge\mathbf{v}_A$ to the lift involves the performance of no work. Thus the rate at which energy exhausts from the nozzle owing to the excitation of incompressible disturbances in the exterior jet flow is just equal to the rate at which work is performed on the exterior fluid by the Reynolds-stress system $-\rho_0 v_i v_j$ of the vortical flow in the rate-of-strain field of the acoustic disturbance. The energy provided in this way is used to generate vorticity at the lip of the nozzle and may thereby maintain a steady system of spatial instability waves on the jet. The latter would be identified with the incipient form of the 'coherent structures' observed by Moore (1977) and others.

A preliminary quantitative estimate of the magnitude of the hydrodynamic attenuation is readily obtained in the limit of small Strouhal number $\omega a/U_J$. Let u_A denote the amplitude of the mean axial component of the fluctuating jet velocity in the nozzle exit plane. Reference to (3.2) and use of the momentum equation (2.6) (with $\boldsymbol{\chi} \equiv 0$) indicates that

$$u_A = -i\mathscr{A}\beta B_I/A\omega, \tag{3.25}$$

the factor $\mathscr{A}/A$ arising from the continuity of flow in the nozzle. The nozzle fluctuations produce a periodic train of axisymmetric vortex rings whose circulation per unit length in the x_1 direction is equal to $U_J + u_A \exp(-i\omega t)$ at the nozzle exit, and which convect downstream at velocity $\frac{1}{2}[U_J + u_A \exp(-i\omega t)]$ (cf. Saffman 1975). If the mean shear layer close to the nozzle exit is not too thick, it follows in a linearized approximation that

$$\boldsymbol{\omega} \wedge \mathbf{v} = u_A U_J \hat{\mathbf{r}}\, \delta(r-a) \exp[-i\omega(t - 2x_1/U_J)], \tag{3.26}$$

where the radial co-ordinate $r = (x_2^2 + x_2^3)^{\frac{1}{2}}$ and $\hat{\mathbf{r}}$ is the corresponding unit vector.

In the case of a cold jet, for which $\rho_1 \equiv \rho_0$, (3.26) is the only contribution to the aerodynamic source vector $\boldsymbol{\chi}$ of (2.5), and its substitution into (3.9) shows that

$$\gamma + i\delta = \frac{iU_J}{A\omega} \oint \nabla F_A \cdot d\mathbf{S} \exp\left(\frac{2i\omega y_1}{U_J}\right), \tag{3.27}$$

where the surface integral is taken over the nominal boundary $r = a$ of the jet. At sufficiently small Strouhal numbers the variation of the exponential in the integrand may be neglected over that portion of the boundary where ∇F_A is significant, and it then follows from the definition of F_A that

$$\gamma = 0, \quad \delta = \mathscr{A}U_J/A\omega, \tag{3.28}$$

a result which illustrates the dependence of the end correction l_H on the hydrodynamic length scale U_J/ω of the fluctuating nozzle flow.

Inserting these values of γ and δ in (3.17), we find that the reflexion coefficient R has the explicit form

$$R = -\left(\frac{1 - (\mathscr{A}/A)(M_J + \frac{1}{4}(k_0 a)^2) + ik_0\lambda}{1 + (\mathscr{A}/A)(M_J + \frac{1}{4}(k_0 a)^2) - ik_0\lambda}\right), \tag{3.29}$$

in which $k_0 = k_1$ for a cold jet. Similarly the energy flux W_T becomes

$$W_T = \frac{W_0(\mathscr{A}/A)\{4M_J + (k_0 a)^2)\}}{(1 + (\mathscr{A}/4A)(4M_J + (k_0 a)^2))^2 + (k_0\lambda)^2}. \tag{3.30}$$

These approximate expressions, which are nonetheless characteristic of the general case, show that in the presence of the mean nozzle flow the limit of long wavelength

($k_0 a$, $k_0 \lambda \to 0$) does not reproduce the classical results $R = -1$ and $W_T = 0$ for reflexion at an open end. Indeed for a sufficiently large area ratio $\mathscr{A}/A$, it is clear that $R \simeq +1$ and $W_T \simeq 4W_0 A/\mathscr{A} M_J$.

3.3. *Exterior flow models*

A precise determination of the complex hydrodynamic end correction $l_H = \gamma + i\delta$, and thence of the energy flux W_T, involves the analysis of a specific modelling of the exterior shear flow. This is facilitated by the assumption that the nozzle possesses a circular cylindrical neck which extends a distance of at least one nozzle exit radius downstream of the contraction, as in figure 1. It may then be asserted that local details of the exterior incompressible flow do not depend critically on upstream variations in nozzle geometry, a hypothesis which is justified by the observation that the back-reaction B_J produces no additional velocity fluctuations in the upstream region. We shall therefore examine shear flows calculated on the basis of pulsatile incompressible flow from a semi-infinite circular cylindrical duct.

Two cases I and II will be discussed. In case I the boundary of the jet is represented by a linearly disturbed vortex sheet. A thorough discussion of this problem for compressible flow has been given by Munt (1977), therefore a statement of the relevant results obtained in the incompressible limit is sufficient for our purposes. Additional details are outlined in the appendix.

In Munt's theory the Kutta condition of finiteness is imposed at the nozzle lip. This is presumably appropriate at the relatively low Strouhal numbers of interest in the present discussion (cf. Bechert & Pfizenmaier 1975*b*); at higher frequencies and correspondingly smaller length scales, sound emerges from the duct without 'feeling' the lip, and propagates along energy-conserving ray paths through the mean shear layer. The shear layer is unstable, however, the instabilities being associated with eigenmodes of oscillation of the jet, and Munt obtains a strictly *causal* solution in which these modes are triggered and sustained by the fluctuating nozzle flow.

Taking the incompressible limit in Munt's theory (see appendix) the hydrodynamic end correction $l_H = \gamma + i\delta$ can be expressed in the following form in case I:

$$\gamma = \frac{\mathscr{A} U_J}{A\omega}(\zeta - \mu), \quad \delta = \frac{\mathscr{A}}{A}\frac{U_J}{\omega}\nu. \tag{3.31a, b}$$

The dimensionless quantities ζ, μ and ν are real functions of the Strouhal number $\omega a/U_J$ and the density ratio ρ_0/ρ_1, and are defined by means of the dispersion function $Z(U_J k/\omega, ka, \rho_0/\rho_1)$ given in equation (A 7) of the appendix, whose zeros determine those axisymmetric incompressible eigenmodes of a circular cylindrical jet which are proportional to $\exp\{i(kx_1 - \omega t)\}$. When ω is real and positive, say, the zero $k = k_I$ of Z corresponding to the spatially growing instability mode of the jet lies in the fourth quadrant of the k plane, and defines μ and ν by

$$\nu + i\mu = \omega/U_J k_I. \tag{3.32}$$

The imaginary part μ tends to zero as $\omega a/U_J \to 0$. The real part ν determines the attenuation of the incident sound due to the excitation of the instability wave on the jet, and

$$\nu \to \begin{cases} 1 & \text{as} \quad \omega a/U_J \to 0, \\ \rho_1/(\rho_1 + \rho_0) & \text{as} \quad \omega a/U_J \to \infty, \end{cases} \tag{3.33}$$

and varies monotonically between these limits for intermediate values of the Strouhal number.

The remaining function ζ is given in terms of Z by

$$\zeta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln |Z(\xi, \xi(\omega a/U_J), \rho_0/\rho_1)|\, d\xi}{\xi^2}, \tag{3.34}$$

and tends to zero with the Strouhal number $\omega a/U_J$.

Pinker & Bryce (1976) and Savkar (1975) suggest that in practice instability waves are not excited at low subsonic Mach numbers. This would be expected to be the case if the width of the shear layer were sufficiently large with respect to the hydrodynamic length scale of the fluctuating flow. We shall therefore examine the opposite extreme to that discussed by Munt (1977), and assume in case II that the effect of the finite width of the shear layer cannot be neglected.

An approximate analytical treatment of this case may be given by noting that in the experiments of Pinker & Bryce (1976) the width of the shear layer at the nozzle exit is substantially greater than the anticipated viscous-controlled width $\sim (\nu_A/\omega)^{\frac{1}{2}}$ of vorticity shed from the nozzle lip (ν_A being the kinematic viscosity, $\sim 1{\cdot}5 \times 10^5\ \mathrm{m}^2/\mathrm{s}$ for air at 300 °K) even at the lowest frequency of interest. Thus if attention is confined to a *cold* jet, in the linearized approximation the aerodynamic source dipole becomes

$$\boldsymbol{\chi} \equiv \boldsymbol{\omega} \wedge \mathbf{v} = \boldsymbol{\omega}_0 \wedge \mathbf{v}' + \boldsymbol{\omega}' \wedge \mathbf{U} + \boldsymbol{\omega}_0 \wedge \mathbf{U}, \tag{3.35}$$

where $\mathbf{v}'$ and $\boldsymbol{\omega}'$ respectively denote the perturbation velocity and vorticity, and $\mathbf{U}$ and $\boldsymbol{\omega}_0$ are the corresponding mean flow quantities. The back-reaction B_J therefore satisfies the following approximate form of (3.3) provided that the length scale of the shed vorticity is small compared with the width of the shear layer:

$$-\nabla^2 B_J = \operatorname{div}(\boldsymbol{\omega}' \wedge \mathbf{U}). \tag{3.36}$$

In this case the shed vorticity is located within the shear layer in a region of effectively uniform mean velocity and convects downstream at this velocity, which is well approximated by one-half of the nominal centre-line jet velocity U_J (Davies, Fisher & Barratt 1963). The vorticity $\boldsymbol{\omega}'$ is assumed to convect along the mean boundary $r = a$ of the jet and its strength is determined via the Kutta condition. The procedure is outlined in the appendix, and leads to the following determination of the hydrodynamic end correction in case II:

$$\gamma = 0, \quad \delta = \mathscr{A} U_J/2A\omega. \tag{3.37}$$

Return now to the general energy flux formula (3.18). We use the above results for the hydrodynamic end correction to obtain the following predictions for the total perturbation energy flux from the nozzle:

Case I. *Vortex-sheet model*:

$$W_T = \frac{W_0\left(\dfrac{\mathscr{A}}{A}\right)\left\{4M_J\nu + \dfrac{\rho_0 c_0}{\rho_1 c_1}(k_0 a)^2\right\}}{\left(1 + \dfrac{\mathscr{A}}{4A}\left[4M_J\nu + \dfrac{\rho_0 c_0}{\rho_1 c_1}(k_0 a)^2\right]\right)^2 + \left(k_1\lambda + \dfrac{\mathscr{A}}{A}M_J(\zeta - \mu)\right)^2}. \tag{3.38}$$

Case II. *Finite width shear layer* (*cold jet*):

$$W_T = \frac{W_0\left(\dfrac{\mathscr{A}}{A}\right)\left\{2M_J + (k_0 a)^2\right\}}{\left(1 + \dfrac{\mathscr{A}}{4A}\left[2M_J + (k_0 a)^2\right]\right)^2 + (k_1\lambda)^2}. \tag{3.39}$$

Observe that the formula (3.38) for case I reduces to the approximation (3.30) in the limit of small Strouhal number (when $\nu \to 1$; ζ, $\mu \to 0$) for a cold jet. Interestingly enough, we also see that, again for a cold jet, the limiting value $\nu \sim 0{\cdot}5$ as $\omega a/U_J \to \infty$ implies that the vortex-sheet model (3.38) reduces to the finite width shear-layer model (3.39).

4. The radiated sound power; comparison with experiment

The substitution of (3.9) and the explicit linearized form of χ_1 into (2.21) enables the sound field to be expressed in the following form:

$$\frac{p}{\rho_0} = \frac{-ik_1\mathscr{A}}{2\pi|\mathbf{x}|}\left\{\left(1+\frac{\beta}{2}(\gamma+i\delta)\left(1-\frac{c_1 A}{c_0\mathscr{A}}\cos\theta\right)\right)B_I\exp\left(-i\omega[t]\right)+\frac{c_1\cos\theta}{2c_0}\int[\omega_0 v_r]\,d^3\mathbf{y}\right\}, \tag{4.1}$$

where v_r is the radial component of the perturbation velocity. In the case of the vortex-sheet model the mean vorticity ω_0 forms a singular distribution on $r = a$, and the integration in (4.1) reduces to the determination of the volume flux through the nominal boundary of the jet. Since compressible effects are unimportant in the nozzle region this flux is equal to the volume flux from the nozzle, and in the approximation of (3.1) the integrated term of (4.1) becomes $\frac{1}{2}B_I(1-R)M_0\cos\theta\exp(-i\omega[t])$. This result would also be expected to be valid to a good approximation for a shear layer of finite width, because the principal contribution to the integral is from the region close to the nozzle where the shear layer is relatively thin. Hence using this, and (3.15*a*) and (3.17) to calculate β, it follows that in the approximation of long wavelength the intensity of the free-space radiation becomes

$$\frac{\langle p^2\rangle}{\rho_0 c_0} = \frac{W_0(\mathscr{A}/A)(\rho_0 c_0/\rho_1 c_1)(k_0 a)^2}{4\pi|\mathbf{x}|^2}\left\{\frac{1+(1+\nu)M_0\cos\theta)^2 + M_0^2(\zeta-\mu)^2\cos^2\theta}{(1+(\mathscr{A}/A)M_J\nu)^2+(\mathscr{A}/A)^2M_J^2(\zeta-\mu)^2}\right\} \quad \text{in case I,} \tag{4.2}$$

$$\frac{\langle p^2\rangle}{\rho_0 c_0} = \frac{W_0(\mathscr{A}/A)(k_0 a)^2}{4\pi|\mathbf{x}|^2}\left\{\frac{1+\frac{3}{2}M_0\cos\theta}{1+\mathscr{A}M_J/2A}\right\}^2 \quad \text{in case II (cold jet).} \tag{4.3}$$

When $\mathscr{A} = A$ (no nozzle contraction) the vortex-sheet case I agrees with the corresponding limiting form of Munt's (1977) exact analysis.

The directivities predicted by these formulae are illustrated in figure 2 and compared with the field shape data of Pinker & Bryce (1976). Those data are for a cold jet with $M_0 \equiv M_J = 0{\cdot}3$, and the curve in case I has been calculated for a Strouhal number $\omega a/U_J = 0{\cdot}8$ which corresponds to the experiment at the Helmholtz number $k_0 a = 0{\cdot}24$. Actually the comparison with experiment is probably relevant only for θ in the range 60°–120°, say, where $\cos^2\theta$ is not particularly significant, since our analysis has systematically neglected terms $O(M_0^2)$ relative to unity. The data have

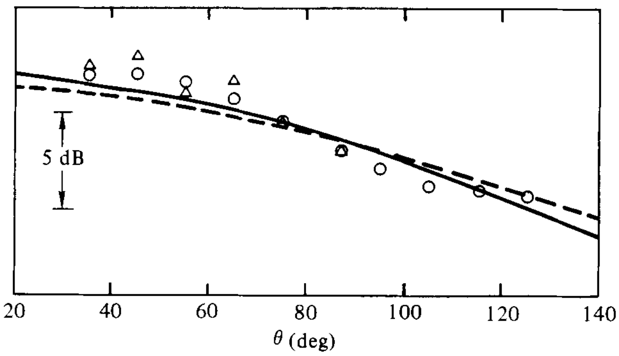

FIGURE 2. Predicted field shape characteristics for $ka = 0{\cdot}24$ and $M_J = 0{\cdot}3$. ——, case I, vortex-sheet model; ---, case II, finite width shear-layer model. The angle θ is measured from the downstream direction of the jet axis, and the experimental points are taken from Pinker & Bryce (1976) at $M_J = 0{\cdot}3$: △, $ka = 0{\cdot}24$; ○, $ka = 0{\cdot}6$.

been matched with the theory at $\theta = 90°$, and for the above range in θ both models exhibit a tolerable representation of the experimental results, although the overall agreement is better for the vortex-sheet model of case I. Note, however, that there exists an absolute difference in the levels predicted by the two models at 90°, caused principally by the presence of the area ratio $\mathscr{A}/A$ in the denominator of each of (4.2) and (4.3). In the Pinker & Bryce experiment $\mathscr{A}/A \simeq 3{\cdot}7$, and this implies that case II exceeds case I by about 2·5 dB at $\theta = 90°$. This is the only way in which the area ratio $\mathscr{A}/A$ influences the radiated sound field, and presumably accounts for the good agreement with experiment of Munt's (1977) field shape predictions based on a circular cylindrical nozzle.

Integration of (4.2) and (4.3) over the surface of a large sphere of radius $|\mathbf{x}|$ centred on the nozzle exit yields the total radiated sound power W_F. The contribution from the dipole component of the field is now $O(M_0^2)$ relative to that of the monopole, and this must be rejected in order to be consistent with our previous approximations. In the limit of long wavelength we then find by comparison with the equations (3.38) and (3.39), giving the energy flux W_T through the nozzle

$$\frac{W_F}{W_T} = \frac{(\rho_0 c_0/\rho_1 c_1)(k_0 a)^2}{4M_J\nu + (\rho_0 c_0/\rho_1 c_1)(k_0 a)^2} \quad \text{for case I,} \tag{4.4}$$

$$\frac{W_F}{W_T} = \frac{(k_0 a)^2}{2M_J + (k_0 a)^2} \quad \text{for case II (cold jet).} \tag{4.5}$$

These results are independent of the area ratio $\mathscr{A}/A$.

Bechert *et al.* (1977) have measured the attenuation $10\log_{10}(W_F/W_T)$ dB in the case of a cold jet over a range of subsonic nozzle exit Mach numbers M_J, the area ratio $\mathscr{A}/A$ being equal to 7·6. Their results are shown in figure 3. Figure 4 compares the predictions (4.4) and (4.5) for cases I and II with the particular low Mach number case $M_J \equiv M_0 = 0{\cdot}3$. The theoretical curves predict identical overestimates of the attenuation at the higher values of ka, for which $\nu \sim 0{\cdot}5$, but of course the compact approximation used in deriving our results would be expected to fail in this region. At lower values of ka the finite shear layer model produces a marginally better agreement with experiment. In any event the agreement with experiment is sufficiently

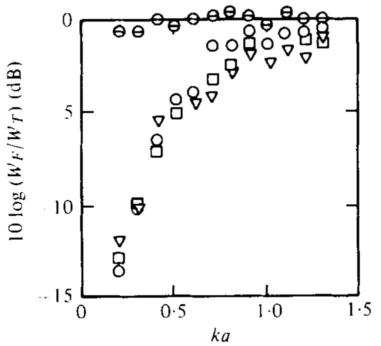

FIGURE 3. Measured ratio of the far-field sound power W_F to the nozzle power flux W_T as a function of the nozzle exit Helmholtz number ka for various values of jet Mach number M_J (Bechert *et al.* 1977). ⊖, $M_J = 0$; ○, $M_J = 0{\cdot}3$; □, $M_J = 0{\cdot}5$; ▽, $M_J = 0{\cdot}7$.

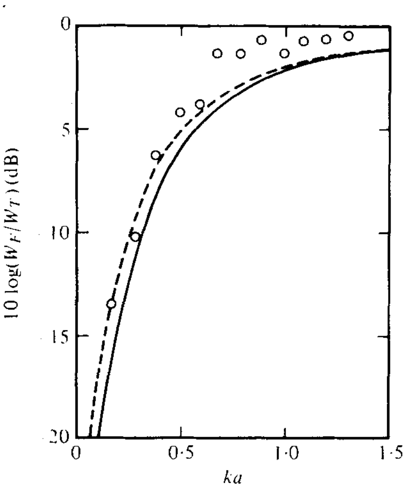

FIGURE 4. Comparison of predicted and measured ratio W_F/W_T as a function of ka for $M_J = 0{\cdot}3$. Experiment: ○, Bechert *et al.* (1977). Theory: ——, case I, vortex-sheet model; ----, case II, finite width shear-layer model.

good to give confidence in the validity of the hydrodynamic attenuation mechanism, and indicates that at low frequencies the details of both the attenuation levels and the radiation directivities are relatively insensitive to the precise modelling of the exterior nozzle flow.

5. Conclusion

The emission of low frequency sound from a jet pipe in the presence of a subsonic nozzle flow involves a transfer of energy from the acoustic wave to essentially incompressible vortex waves on the jet. This produces a net attenuation in the transmitted sound which is not compensated by the coherent and/or broad-band aerodynamic sound subsequently radiated by the vortex motions. At low Mach number and low Helmholtz number ka the predicted attenuation and the field shape of the radiated sound do not depend critically on the details of the theoretical modelling of the exterior flow, although a vortex-sheet model tends to predict a slightly lower overall level of radiation than one which incorporates the effects of the finite width of the shear layer. This supports the view that, as far as the interaction with the nozzle is concerned, the question of whether or not the exterior flow is stable is quite irrelevant, because the principal interaction occurs within one hydrodynamic length scale from the lip of the nozzle. The Kutta condition plays a much more significant role inasmuch as shed vorticity both provides the vehicle by which hydrodynamic energy is conveyed downstream, and also, through its interaction with the nozzle, is responsible for the production of the aerodynamic component of the radiated sound and, therefore, for the characteristic field shape variations associated with the mean nozzle flow.

In the case of the jet pipe of an aeroengine, Bechert *et al.* (1977) point out that the large hydrodynamic attenuation observed in their experiment at low frequencies would correspond to the lower end of the audible range (50–100 Hz). Effective attenuation could be achieved at higher frequencies, however, by making use of multitube nozzles. Dean & Tester (1975) have already exploited this silencing mechanism by means of a bias air flow through an acoustic wall liner, an expedient originally proposed for this purpose by Barthel (1958).

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Appendix

A 1. *Compact Green's function for an axisymmetric nozzle*

In the absence of a mean flow Ffowcs Williams & Howe (1975) have given the following expression for the compact approximation to the Green's function for an axisymmetric nozzle of the type shown in figure 1:

$$G(\mathbf{x},\mathbf{y},t,\tau) = \frac{1}{4\pi|\mathbf{x}|}\,\delta\left\{t-\tau-\frac{(|\mathbf{x}-\mathbf{K}(\mathbf{y})|-(c_0/c_1)\,F_A(\mathbf{y}))}{c_0}\right\}. \qquad \text{(A 1)}$$

This is suitable for treating aerodynamic noise problems in which the characteristic wavelength of the sound is large compared with the nozzle radius. The observer is located in free space at the far-field point $\mathbf{x}$, c_0 is the sound speed in the ambient medium and c_1 that upstream of the nozzle exit.

The functions $F_A(\mathbf{y})$ and $\mathbf{K}(\mathbf{y})$ are harmonic, and satisfy the normal velocity condition $\mathbf{n}\cdot\nabla(F_A,\mathbf{K}) = 0$ on the walls of the rigid nozzle, $\mathbf{n}$ being the unit normal. In particular, $F_A(\mathbf{y})$ is the potential of an axisymmetric incompressible nozzle flow, and, taking the co-ordinate origin in the centre of the nozzle exit plane, as in the main text, is normalized such that:

(i) for $|\mathbf{y}| \gg a$ in free space

$$F_A(\mathbf{y}) \simeq -\frac{\mathscr{A}}{4\pi|\mathbf{y}|};$$

(ii) within the nozzle in the vicinity of the point N of figure 1,

$$F_A(\mathbf{y}) \simeq \frac{\mathscr{A}}{A}(y_1 - l_0),$$

$l_0 = 0{\cdot}6133a$ being the 'end correction' for a semi-infinite circular cylindrical pipe (Noble 1958, p. 138);

(iii) upstream of the nozzle contraction

$$F_A(\mathbf{y}) \simeq y_1 - \lambda,$$

where λ is the effective, geometric nozzle end correction given approximately by

$$\lambda = \frac{\mathscr{A}}{A}\left\{l_0 + \left(\Delta + \frac{L}{2}\right)\left(\frac{\mathscr{A} - A}{\mathscr{A}}\right)\right\}, \tag{A 2}$$

Δ being the length of the neck of the nozzle, and L the axial distance over which the contraction occurs (Rayleigh 1945, § 308).

The vector function $\mathbf{K}(\mathbf{y})$ has the following properties:

(i) for $|\mathbf{y}| \gg a$ in free space

$$\mathbf{K}(\mathbf{y}) \simeq \mathbf{y};$$

(ii) for $|y_1| \gg a$ in the nozzle

$$\mathbf{K}(\mathbf{y}) \simeq \text{constant};$$

(iii) in the vicinity of the nozzle exit

$$\mathbf{K}(\mathbf{y}) = \left(y_1 - \frac{A}{\mathscr{A}}F_A(\mathbf{y}), F_B(\mathbf{y}), F_C(\mathbf{y})\right);$$

the precise forms of the potential functions F_B and F_C are not required in applications to axisymmetric source distributions.

The compact approximation (A 1) is valid asymptotically in the sense that, for source locations $\mathbf{y}$ well within an acoustic wavelength of the nozzle, the multipole expansion obtained by developing the δ-function in a retarded time series of the form $\Sigma A_n(\mathbf{x}, \mathbf{y})\,\delta^{(n)}(t - \tau - |\mathbf{x}|/c_0)$ is correct for $n = 0, 1$. At low Mach numbers these terms describe the principal interaction of the shear-layer sources with the nozzle. The method of derivation of (A 1) is described by Howe (1975, appendix), and by Ffowcs Williams & Howe (1975) for closely analogous problems.

In the presence of a low Mach number nozzle flow, in which the characteristic wavelengths remain large compared with the nozzle diameter, (A 1) must be modified to take account of the convection of sound by the mean flow. Since $F_A \simeq y_1$, $\mathbf{K} \simeq$ constant in the upstream region, convection of the 'imploding' wave results in the presence of a Doppler factor $(1 + M)$, M being the upstream Mach number, and the modified form of (A 1) is

$$G(\mathbf{x}, \mathbf{y}, t, \tau) = \frac{1}{4\pi|\mathbf{x}|} = \delta\left\{t - \tau - \frac{|\mathbf{x} - \mathbf{K}(\mathbf{y})| - (c_0/c_1)\,F_A(\mathbf{y})/(1 + M)}{c_0}\right\}. \tag{A 3}$$

A 2. *Incompressible pulsatile nozzle flow*

The potential $\Psi(\mathbf{x})\exp(-i\omega t) \equiv [F_A(\mathbf{x}) + F_J(\mathbf{x})]\exp(-i\omega t)$ which describes incompressible pulsatile flow in the downstream portion of the nozzle in the presence of a mean flow may be estimated from the corresponding solution for a semi-infinite circular cylindrical duct. In the case in which the shear layer of the exterior flow is modelled by a linearly disturbed vortex sheet, the Wiener–Hopf procedure described by Munt (1977) yields in the incompressible limit the following solution when the Kutta condition is imposed:

$$\Psi(\mathbf{x}) = \lim_{\epsilon \to +0} \frac{\mathscr{A}}{A}\left\{-\frac{\exp(-\epsilon x_1)}{2\epsilon} + \frac{a}{4\pi}\int_{-\infty}^{\infty} \frac{(\omega - i\epsilon U_J)\mathscr{F}(k,r)\,K_+(i\epsilon)\,K_-(k)}{|k|\,(\omega - U_J k)\,Z_+(i\epsilon)\,Z_-(k)} \exp(ikx_1)\,dk\right\}, \tag{A 4}$$

where
$$\mathscr{F}(k,r) = \begin{cases} I_0(|k|r)/I_1(|k|a) & (r < a), \\ -\omega K_0(|k|r)/(\omega - U_J k)\,K_1(|k|a) & (r > a), \end{cases} \tag{A 5}$$

and the first term in the brace brackets of (A 4) is omitted when $r > a$. Here and elsewhere $|k| = (k^2+\epsilon^2)^{\frac{1}{2}}$, and I_n and K_n are modified Bessel functions of order n (Abramowitz & Stegun 1964, p. 374).

The various quantities appearing in these formulae are defined as follows. A function $f(k)$ which is regular and non-zero on the real k axis defines functions $f_\pm(k)$ respectively regular and non-zero in $\operatorname{Im} k \gtrless 0$ by means of

$$f_\pm(k) = \exp\left\{\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln f(\xi)}{\xi - k}\,d\xi\right\}, \tag{A 6}$$

provided that the integral exists in an appropriate sense (Noble 1958, p. 13).

The functions K and Z are given by

$$K(k) = 2I_1(|k|a)\,K_1(|k|a), \tag{A 7a}$$

$$\left.\begin{aligned} Z(k) &\equiv Z\left(\frac{U_J k}{\omega}, ka, \frac{\rho_0}{\rho_1}\right), \\ &= |k|a\left\{\frac{\rho_0}{\rho_1}\cdot\frac{I_1(|k|a)\,K_0(|k|a)}{\left(1-\dfrac{U_J k}{\omega}\right)^2} + I_0(|k|a)\,K_1(|k|a)\right\}. \end{aligned}\right\} \tag{A 7b}$$

As $x_1 \to -\infty$ within the circular cylindrical duct the principal contribution to the integral in (A 4) is from a simple pole at $k = -i\epsilon$. This yields the approximate form of Ψ close to the point N of the nozzle of figure 1, viz:

$$\Psi \simeq \lim_{\epsilon \to +0} \frac{\mathscr{A}}{A}\left\{-\frac{\exp(-\epsilon x_1)}{2\epsilon} + \left(\frac{\omega - i\epsilon U_J}{\omega + i\epsilon U_J}\right)\frac{K_+(i\epsilon)\,K_-(-i\epsilon)}{Z_+(i\epsilon)\,Z_-(-i\epsilon)}\frac{\exp(\epsilon x_1)}{2\epsilon}\right\}. \tag{A 8}$$

Now as $\epsilon \to 0$
$$K_+(i\epsilon)\,K_-(-i\epsilon) \simeq 1 - 2\epsilon l_0, \tag{A 9}$$

where $l_0 = 0{\cdot}6133a$ is the end correction of a circular cylindrical pipe (Noble 1958, p. 138).

The solution (A 4) will satisfy the causality condition, i.e. the condition that the fluctuations in the exterior flow are a *consequence* of the pulsating nozzle flow, provided that it is regular in an upper complex ω plane (Lighthill 1960). For $\operatorname{Im}\omega \sim +\infty$ the dispersion function $Z(k) \to \frac{1}{2}$ as $k \to \pm\infty$ on the real axis and

$$Z_+(i\epsilon)\,Z_-(-i\epsilon) = \exp\left\{\frac{\epsilon}{\pi}\int_{-\infty}^{\infty} \frac{\ln Z(k)\,dk}{k^2+\epsilon^2}\right\}. \tag{A 10}$$

The result for real ω is obtained by analytic continuation, and as ω approaches the positive real axis, say, the zero $k = k_I$ of the dispersion function $Z(k)$ which corresponds to the instability mode of the semi-infinite jet crosses the real k axis from the first quadrant into the fourth quadrant. Deforming the contour in (A 10) to take account of this, and comparing the result with the integral along the real k axis, we find that for real positive ω

$$Z_+(i\epsilon)\,Z_-(-i\epsilon) \simeq \exp\left\{\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\ln Z(k)\,dk}{k^2}+\frac{2i\epsilon}{k_I}\right\}. \tag{A 11}$$

The integral here is split into real and imaginary parts by noting that as $\mathrm{Im}\,\omega \to +0$ the argument of $Z(k)$ decreases discontinuously by 2π as k increases through $k = \omega/U_J$, and in this way we find that for small ϵ

$$Z_+(i\epsilon)\,Z_-(-i\epsilon) \simeq \exp\left\{\frac{\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\ln|Z(k)|\,dk}{k^2}-\frac{2i\epsilon U_J}{\omega}+\frac{2i\epsilon}{k_I}\right\}. \tag{A 12}$$

Substitution of (A 9) and (A 12) into (A 8) gives the limiting value

$$\Psi = \frac{\mathscr{A}}{A}\left\{x_1-l_0-\frac{U_J}{\omega}(\zeta-\mu)-i\frac{U_J}{\omega}\nu\right\}, \tag{A 13}$$

where ζ, μ and ν are defined in (3.32) and (3.34). Reference to the defining properties of $F_A(\mathbf{x})$ given above then shows that

$$F_J(\mathbf{x}) \to -\frac{\mathscr{A}}{A}\frac{U_J}{\omega}[(\zeta-\mu)+i\nu], \tag{A 14}$$

upstream of the nozzle contraction, which immediately leads to (3.31*a*, *b*).

A 3. *The case of a finite width shear layer*

To solve (3.36) first set in the linearized approximation

$$\boldsymbol{\omega}'\wedge\mathbf{U} = \tfrac{1}{2}U_J\sigma\hat{\mathbf{r}}\exp\left[-i\omega\left(t-\frac{2x_1}{U_J}\right)\right]\delta(r-a), \tag{A 15}$$

and solve (3.36) by the Wiener–Hopf procedure to give

$$B_J = \lim_{\epsilon\to+0}\frac{a\omega\sigma}{4\pi i}K_+\left(\frac{2\omega}{U_J}\right)\int_{-\infty}^{\infty}\frac{(k-i\epsilon)\,K_-(k)\,I_0(|k|r)}{|k|I_1(|k|a)\,(k-2\omega/U_J)}\exp(ikx_1)\,dk \tag{A 16}$$

for $r \leqq a$.

Write $B_A = \beta B_I F_A(\mathbf{x})$. The circulation density σ of the shed vorticity is chosen to ensure that B_A+B_J satisfies the Kutta condition at the nozzle lip. An integral expression for B_A is obtained by setting $U_J \equiv 0$ in (A 4), and by considering the behaviour of the integrands in (A 4) and (A 16) as $k \to \infty$ it follows in the usual way (Jones 1972) that

$$\sigma = \frac{-i\mathscr{A}\beta B_I}{\omega A K_+(2\omega/U_J)}. \tag{A 17}$$

This may be used in (A 6) to determine the upstream limiting form of B_J and thence to give (3.37).

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Washington: Nat. Bur. Stand.

BARTHEL, VON F. 1958 Untersuchungen uber nichtlineare Helmholtzresonatoren. *Frequenz* **12**, 1–11.

BECHERT, D., MICHEL, U. & PFIZENMAIER, E. 1977 Experiments on the transmission of sound through jets, *A.I.A.A. Paper* no. 77–1278.

BECHERT, D. & PFIZENMAIER, E. 1975*a* On the amplification of broad band jet noise by a pure tone excitation. *J. Sound Vib.* **43**, 581–587.

BECHERT, D. & PFIZENMAIER, E. 1975*b* Optical compensation measurements on the unsteady exit condition at a nozzle discharge edge. *J. Fluid Mech.* **71**, 123–144.

BLOKHINTSEV, D. I. 1946 Acoustics of a nonhomogeneous moving medium. *N.A.C.A. Tech. Memo.* no. 1399.

BROWN, G. B. 1935 On vortex motion in gaseous jets and the origin of their sensitivity to sound. *Proc. Phys. Soc.* **47**, 703.

CRIGHTON, D. G. 1972 The excess noise field of subsonic jets. *Aero. Res. Counc. Rep.* no. 33 714 N. 781.

CROW, S. C. 1972 Acoustic gain of a turbulent jet. *Meeting Div. Fluid Dyn. Am. Phys. Soc., Univ. Colorado*, paper IEG.

DAVIES, P. O. A. L., FISHER, M. J. & BARRATT, M. J. 1963 The characteristics of the turbulence in the mixing region of a round jet. *J. Fluid Mech.* **15**, 337–367.

DEAN, P. D. & TESTER, B. J. 1975 Duct wall impedance control as an advanced concept for acoustic suppression. *N.A.S.A. Contractor Rep.* no. 134998.

FFOWCS WILLIAMS, J. E. & HOWE, M. S. 1975 The generation of sound by density inhomogeneities in low Mach number nozzle flows. *J. Fluid Mech.* **70**, 605–622.

FREYMUTH, P. 1966 On transition in a separated laminar boundary layer. *J. Fluid Mech.* **25**, 683–704.

GEREND, R. P., KUMASAKA, H. P. & ROUNDHILL, J. P. 1973 Core engine noise, *A.I.A.A. Paper* no. 73–1027.

HOWE, M. S. 1975 Contributions to the theory of aerodynamic sound, with application to excess jet noise and the theory of the flute. *J. Fluid Mech.* **71**, 625–673.

JONES, D. S. 1972 Aerodynamic sound due to a source near a half-plane. *J. Inst. Math. Appl.* **9**, 114–122.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics.* Pergamon.

LIEPMANN, H. W. & ROSHKO, A. 1957 *Elements of Gasdynamics.* Wiley.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. I. General theory. *Proc. Roy. Soc.* A**221**, 564–587.

LIGHTHILL, M. J. 1960 Studies on magneto–hydrodynamic waves and other anisotropic wave motions. *Phil. Trans. Roy. Soc.* A**252**, 397–430.

MOORE, C. J. 1977 The role of shear-layer instability waves in jet exhaust noise. *J. Fluid Mech.* **80**, 321–367.

MUNT, R. M. 1977 The interaction of sound with a subsonic jet issuing from a semi-infinite cylindrical pipe. *J. Fluid Mech.* **83**, 609–640.

NOBLE, B. 1958 *Methods based on the Wiener–Hopf Technique.* Pergamon.

PINKER, R. A. & BRYCE, W. D. 1976 The radiation of plane wave duct noise from a jet exhaust, statically and in flight. *U.K. Nat. Gas Turbine Est. Note* no. NT-1024.

RAYLEIGH, LORD 1945 *The Theory of Sound.* Dover.

SAFFMAN, P. G. 1975 On the formation of vortex rings. *Stud. Appl. Math.* **54**, 261–268.

SAVKAR, S. D. 1975 Radiation of cylindrical duct acoustic modes with flow mismatch. *J. Sound Vib.* **42**, 363–386.

STRATTON, J. A. 1941 *Electromagnetic Theory.* McGraw-Hill.